JOURNAL OF COMPUTATIONAL PHYSICS 12, 24–48 (1973)

An Integro-Differential Equation Technique for the Time-Domain Analysis of Thin Wire Structures. I. The Numerical Method*

E. K. MILLER

University of California, Lawrence Livermore Laboratory, Livermore, California 94550

A. J. POGGIO

Cornell Aeronautical Laboratory, Buffalo, New York 14221

AND

G. J. BURKE

MB Associates, San Ramon, California 94583

Received October 12, 1971

An integral equation is developed for determining the time-dependent current distribution on a wire structure excited by an arbitrary time-varying electric field. The subsectional collocation form of the method of moments is used to reduce this integral equation to a form that can be evaluated on a digital computer as an initial value problem. A Lagrangian interpolation scheme is introduced so that the dependent variables can be accurately evaluated at any point in the spacetime cone; thus, only mild restrictions on the space and time sample density are required. The integral equation relating present values of the current to previously computed values is presented in a form that can be directly converted into a computer code. Expressions are developed for the computer time and the relative advantages of time-domain and frequency-domain calculations are discussed, providing impetus for analyses in the time domain in certain cases.

Part II of this paper will present well-validated numerical results obtained using the technique described.

INTRODUCTION

Interest in determining the transient response of antennas and scatterers has grown steadily in recent years. Impetus for this increased attention to short-pulse electromagnetics has been supplied from a variety of developments, not the least

* Worked performed under the auspices of the U.S. Atomic Energy Commission.

of which are the advances in experimental techniques and hardware. As a result, there has been a resurgence in analytical efforts devoted to the development of solution techniques for the direct time-domain treatment of electromagnetic problems and the corresponding time-domain concepts analogous to gain, efficiency, etc., that have proved so useful in characterizing the frequency-domain properties of antennas and scatterers.

This paper is concerned with obtaining time-domain solutions for radiation and scattering problems. Particular attention will be directed to the treatment of wire antennas and scatterers, using the time-dependent electric-field integral equation. The general approach is similar to that used by Bennett and Weeks [1, 2] for the analysis of solid surface objects by the time-dependent magnetic-field integral equation. Features of the present treatment are also common to that described by Sayre and Harrington [3], Sayre [4], and Auckenthaler and Bennett [5], an important difference however being the apparently more general solution process employed here, which leads to improved numerical accuracy and scope of application. Further work has been reported by Bennett, Auchenthaler, and De Lorenzo [6]. Several examples of actual calculations will be presented in Part II to demonstrate various facets of the solution procedure.

The theoretical concepts pertinent to a time-domain formulation are outlined in the following section, while the reduction of the integral equation to a form suitable for numerical calculations is described under "Numerical Solution." Also discussed in that section are the relative computational advantages of frequency-domain and time-domain analyses. Application of the technique to various antenna and scatterer geometries will be taken up in Part II.

FORMULATION

Alternative methods are available for developing a time-dependent integral equation. One obvious approach is the Fourier transform to the time domain of a frequency-domain version of the equation sought. Or one might return to the time-dependent Maxwell's equations and proceed to derive directly in the time domain the particular equation type of interest. Since this approach will generally show in a clearer way the manner in which the various terms in the final equation originate, it is the one we follow here. It is not necessarily, however, the most efficient nor straightforward method to pursue, a fact that is illustrated by the derivation of the time-dependent vector-potential integral equations for thin wires given by Mei [7], who used a formulation in the time domain, and Poggio [8], who Fourier transformed the frequency-domain equations.

At the outset, it is necessary that the type of structure be identified to which the analysis is to be devoted. It has been determined, for example, that integral equations of the magnetic type (the magnetic field is the forcing function) are generally better suited to the treatment of solid surfaces [9]. The electric-field integral equation, on the other hand, is a better choice for the analysis of wires and thin-walled open structures. Although neither of these statements is more than a guideline, and examples can be found that contradict them (Lin and Mei [10], for example, successfully treated a wire antenna with the magnetic type of integral equation), they provide a fairly useful demarcation between the most frequently encountered applications of these integral-equation types.

Since our interest is primarily the analysis of wire antennas and scatterers, we chose the electric-field integral equation. While an equation based on the vector potential formulation would perhaps serve as well for some problems, it appears from a practical viewpoint that numerical solution of the electric-field equation is more conveniently obtained and is applicable without reformulation to a wider class of problem. This has been found to be the case in frequency-domain applications for which the vector potential method, although being possibly superior for problems such as the expanding-wire conical antenna, appears otherwise to offer no obvious advantages in solution efficiency or accuracy over the electric-field approach.

The time-dependent Maxwell's equations furnish the starting point of our deviation:

$$\nabla \times \bar{E} = -\mu_0 \frac{\partial}{\partial t} \bar{H}$$

$$\nabla \times \bar{H} = \epsilon_0 \frac{\partial}{\partial t} \bar{E} + \bar{J}$$

$$\nabla \cdot \bar{J} + \frac{\partial}{\partial t} \rho = 0 \qquad \nabla \cdot \bar{H} = 0, \tag{1}$$

where μ_0 and ϵ_0 represent the permeability and permittivity of free space, $\bar{E}$ and $\bar{H}$ are the electric and magnetic fields, and $\bar{J}$ and ρ are the volume current and charge densities, respectively.

The derivation of the desired integral equation may proceed by the vector Green's identity, which with (1) leads to an integral equation relating $\bar{J}$ and $\bar{E}$ on the surface of the body under consideration [9]. By invoking the usual further approximations that the structure is wirelike with a circular cross section small compared with the wavelength of the highest significant frequency component of the excitation spectrum, etc., the surface integral over surface current density can be reduced to a line integral along the structure's periphery. This procedure, while rigorous, leads to the same result as obtained by adopting at the outset the assumptions inherent in the thin-wire approximation. The latter approach, while somewhat heuristic, is more direct and is the one we employ here.

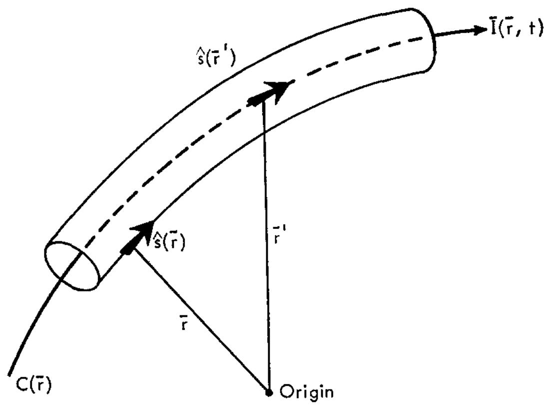

FIG. 1. Geometry for thin-wire electric-field integral equation.

Consider a filamentary current $\bar{I}(\bar{r}, t)$ flowing on the path $C(\bar{r})$ (see Fig. 1) along which the length variable is s. The electric field it produces is determined by [11]

$$\bar{E}(\bar{r}, t) = -\nabla\Phi(\bar{r}, t) - \frac{\partial}{\partial t}\bar{A}(\bar{r}, t), \tag{2}$$

where

$$\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi}\int_{C(\bar{r})} \frac{\bar{I}(\bar{r}', t - R/c)}{R}\, ds' \tag{3}$$

and

$$\Phi(\bar{r}, t) = \frac{1}{4\pi\epsilon_0}\int_{C(\bar{r})} \frac{q(\bar{r}', t - R/c)}{R}\, ds', \tag{4}$$

where $s = s(\bar{r})$, $s' = s(r')$, $ds' = ds(\bar{r}')$, $R = |\bar{R}| = \bar{r} - \bar{r}'|$ and the unprimed coordinates $\bar{r}$ and t denote the observation point location and the primed coordinates $\bar{r}'$ and $t' = t - R/c$ the source location. The differential operators in (2) are with respect to the observation coordinates.

If we let $\hat{s} = \hat{s}(\bar{r})$ and $\hat{s}' = \hat{s}(\bar{r}')$ be the unit tangent vectors to $C(\bar{r})$ at $\bar{r}$ and $\bar{r}'$, then the required terms in (2) can be written

$$\frac{\partial}{\partial t}\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi}\int_{C(\bar{r})} \frac{\hat{s}'}{R}\frac{\partial}{\partial t'} I(\bar{r}', t')\, ds' \tag{5}$$

and

$$\nabla\Phi(\bar{r}, t) = \frac{1}{4\pi\epsilon_0}\int_{C(\bar{r})} \left[-q(\bar{r}', t')\frac{\bar{R}}{R^3} + \frac{\bar{R}}{R^2 c}\frac{\partial}{\partial s'} I(\bar{r}', t')\right] ds', \tag{6}$$

where use has been made of

$$\nabla q(\bar{r}', t') = \frac{\partial}{\partial s'} I(\bar{r}', t') \frac{\bar{R}}{Rc}$$

and

$$\frac{\partial}{\partial s'} I(\bar{r}', t') = -\frac{\partial}{\partial t'} q(\bar{r}', t').$$

Upon noting that $I(\bar{r}', t') \equiv I(s', t')$ and $q(\bar{r}', t') \equiv q(s', t')$ and combining (5) and (6) with (2), one obtains the integral representation for the electric field due to a filamentary current:

$$\bar{E}(\bar{r}, t) = -\frac{\mu_0}{4\pi} \int_{C(\bar{r})} \left[\frac{\hat{s}'}{R} \frac{\partial}{\partial t'} I(s', t') + c \frac{\bar{R}}{R^2} \frac{\partial}{\partial s'} I(s', t') - c^2 \frac{\bar{R}}{R^3} q(s', t')\right] ds'. \tag{7}$$

Equation (7) is valid for all space and time excluding the immediate source region, which in actuality is a conductor of nonzero cross section, so that $|\, \bar{r} - \bar{r}' \,| \geqslant a(\bar{r}')$, the wire radius at $\bar{r}'$. Following the standard thin-wire approach (12, 13) we assume that $I(s', t')$ and $q(s', t')$ are confined to the conductor axis (assumed circular in cross section) and that the boundary condition on the tangential electric field at the conductor surface is known. For convenience we assume a perfect conductor, so that $\hat{s} \cdot (\bar{E} + \bar{E}^A) = 0$, where $\bar{E}^A$ is the applied field that induces the current I that generates the field $\bar{E}$.

By applying the boundary condition on the tangential electric field at the conductor surface to Eq. (7) one obtains the time-dependent electric field integral equation for thin conducting wires in the form

$$\begin{aligned} \hat{s} \cdot \bar{E}^A(\bar{r}, t) = \frac{\mu_0}{4\pi} \int_{C(\bar{r})} \Big[& \frac{\hat{s} \cdot \hat{s}'}{R} \frac{\partial}{\partial t'} I(s', t') + c \frac{\hat{s} \cdot R}{R^2} \frac{\partial}{\partial s'} I(s', t') \\ & - c^2 \frac{\hat{s} \cdot \bar{R}}{R^3} q(s', t') \Big] ds' \qquad \bar{r} \in C(\bar{r}) + a(\bar{r}) \end{aligned} \tag{8}$$

where q can be expressed in terms of I as

$$q(s', t') = -\int_{-\infty}^{t'} \frac{\partial}{\partial s'} I(s', \tau)\, d\tau,$$

and $a(\bar{r})$ denotes the wire radius at point $\bar{r}$. Since the integration path in Eq. (8) is along $C(\bar{r})$ while the field evaluation path is displaced by the wire radius, it is always true that $R > 0$ and the integral in Eq. (8) thus has no singularity. This displacement of the observation and source locations by the wire radius is the essence of the thin wire approximation. Equation (8) is the integral equation whose solution is the topic of the following section.

NUMERICAL SOLUTION

The time-dependent electric field integral equation (8) has the causality property explicitly indicated by the retarded time dependence of the source terms in the integrand. Hence the field at point $\bar{r}$ on the wire is affected by the current and charge at $\bar{r}'$ only after a time delay R/c. This interaction time delay allows the solution of (8) as an initial-value problem without the matrix inversion normally required of the corresponding frequency-domain moment-method approach. As will be found below, once the necessary preliminary calculations pertaining to structure geometry have been performed, the numerical solution of (8) may proceed in a simple time-stepping procedure [1] since at each time step the unknown current and charge at a given point are expressed in terms of already-solved current and charge values and the known incident field. Actually, the truth of this statement is subject to certain limitations on the spacetime sample density used in the approximation of the integral equation, but the basic idea is certainly correct. When for example, $c\,\delta t \leqslant \Delta R$ (as used in [1]), with δt and ΔR the time and space sample intervals, then due to causality, the current and charge at a given spacetime sample point are determined entirely by prior-time current-charge values and the present source field value. However, if $c\,\delta t > \Delta R$ is allowed due to cusps in the structure geometry or to other considerations, then the current and charge at a particular spacetime point will be affected by spatially adjacent current-charge values within the same time step, as discussed further below.

Current Expansion

We attempt to solve (8) using subsectional collocation, and for simplicity we restrict our attention to singly connected wire structures. Extension of the procedure to multiple-junction structures should be reasonably straightforward; this has been found to be true of the frequency-domain version of (8) [14].

Subsectional collocation is essentially the process of (1) segmenting the structure (or domain of the integral operator) into a number of subsections (or segments) whose union may approximately or exactly represent the whole, (2) functionally expression (or expanding) the unknown in some suitable form on each of the segments, and (3) enforcing the integral equation in a pointwise manner over the structure (the range of the integral operator). The sectioned structure may not exactly represent the original if a series of straight wire segments is used to model a curved wire. Since the unknown current is dependent upon both space and time, an expansion of the current in time is required in addition to the usual space expansion associated with a frequency-domain formulation.

The actual current spacetime variation can be approximated by

$$I(\bar{r}', t') = I(s', t') \approx \sum_{i=1}^{N_S} \sum_{j=1}^{\infty} I_{ij}(s' - s_i, t' - t_j)\, U(s' - s_i)\, V(t' - t_j), \tag{9}$$

where $I_{ij}(0, 0) = I_{ij}$ is the current value at the center of the i-th space segment (total of N_S) and the j-th time interval (in practice a finite number N_T) and where

$$U(s' - s_i) = \begin{cases} 1, & |s' - s_i| \leqslant \Delta_i/2 \\ 0, & \text{otherwise} \end{cases}$$

$$V(t' - t_j) = \begin{cases} 1, & |t' - t_j| \leqslant \delta_j/2 \\ 0, & \text{otherwise} \end{cases}$$

with Δ_i and δ_j the lengths of space segment i and time interval j centered at s_i and t_j, respectively. The functional dependence of $I_{ij}(s_i'', t_j'')(s_i'' = s' - s_i;$ $t_i'' = t' - t_j)$ upon s_i'' and t_j'' within the sample area ij is determined by the interpolation method chosen. In addition, it is obvious that if $I_{ij}(s_i'', t_j'')$ were the actual current variation within each of the $N_S N_T$ sample areas, the right side of (9) would be exact. The object is to make the approximate equality true to the desired accuracy with the minimum of overall computational effort.

One of the simplest forms for $I_{ij}(s_i'', t_j'')$ to use is the so-called pulse approximation, whereby the function is considered to be a constant within each space segment and time interval. This approach was used by Sayre and Harrington [3] because Harrington [15] had previously used it with good results for frequency-domain calculations. If it is desired, however, to apply the integral equation to structures with bends, sharp curves, or other similar features, this simple pulse expansion will no longer be satisfactory. The reason is that time and space interpolation between sampled values of the unknown (for accurate modeling of the mutual interactions of the structure segments) requires a differentiable expansion function having nonzero derivatives.

A sinusoidal current expansion consisting of constant, sine, and cosine terms has been found to be well suited, in terms of efficiency and accuracy, to the frequency-domain electric-field integral equation. This expansion is equivalent, for segments small compared with the wavelength, to a second-order polynomial or quadratic expansion. Thus we choose a quadratic expression for the current and charge in our subsectional collocation solution of (8).

With the adoption of a specific quadratic current expansion, the functional form of $I_{ij}(s_i'', t_j'')$ may be derived. Various alternatives are available for quadratic interpolation in two dimensions, such as the five-point and nine-point methods.

The nine-point method appears more suitable, since it will provide a more general spacetime representation of the current behavior. This is especially the case since an accurate current expansion is required near the edges of the ij-th spacetime sample area if the generally unrestricted spacetime sample densities required to handle structures with sharp bends, etc., are to be realized.

Because we want to avoid interpolation into the future, the current at time step j will be interpolated backwards to time steps $j-1$ and $j-2$ when $R/[c(t_j - t_{j-1})] = \Delta R < 0.5$. Otherwise, the interpolation in time will be from time step j to $j+1$ and $j-1$. The space interpolation will similarly be from segment i to $i+1$ and $i-1$ (with interpolation to zero at the wire ends). Thus we can put $I_{ij}(s_i'', t_j'')$ in the form of a Lagrangian interpolation function in two dimensions

$$I_{ij}(s_i'', t_j'') = \sum_{l=-1}^{l=+1} \sum_{m=n}^{m=n+2} B_{ij}^{(l,m)} I_{i+l,j+m}, \tag{10}$$

where

$$\begin{aligned} B_{ij}^{(l,m)} &= \prod_{p=-1}^{p-1}{}^{l} \prod_{q=n}^{q=n+2}{}^{m} \frac{(s' - s_{i+p})(t' - t_{j+q})}{(s_{i+l} - s_{i+p})(t_{j+m} - t_{j+q})} \\ &= \prod_{p=-1}^{p=+1}{}^{l} \prod_{q=n}^{q=n+2}{}^{m} \frac{(s_i'' + s_i - s_{i+p})(t_j'' + t_j - t_{j+q})}{(s_{i+l} - s_{i+p})(t_{j+m} - t_{j+q})} \end{aligned} \tag{11}$$

where $| s_i'' | \leqslant \Delta_i/2$, $| t_j'' | \leqslant \delta_j/2$, $n = -2$ for $\Delta R < 0.5$ and -1 otherwise, and

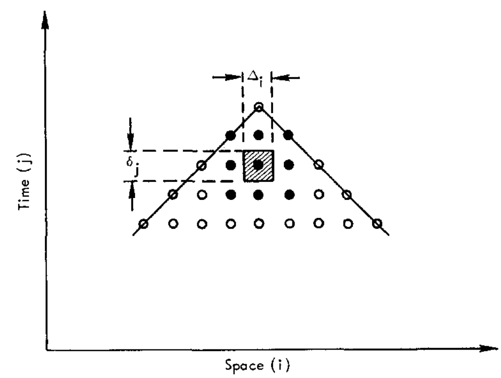

FIG. 2. The spacetime diagram. Each circle represents a sample point; the solid lines separate the past and the future. The ij-th interval is indicated by the shaded region, and the functional dependence of current and charge in that interval is related to the values at net points corresponding to the solid circles.

the superscripts l and m on the product symbols denote that the terms $p = l$, $q = m$ are omitted from (11). Equations (10) and (11) amount to a basis-function expansion of I_{ij} in terms of quadratic space-time surfaces $B_{ij}^{(l,m)}$ having unit value at $s' = s_l$, $t' = t_m$ and zero values at the centers of the eight other space-time samples that are centered about the ij'th sample area. An illustration of the inter-relationship between the space-time sample points is provided in Fig. 2.

When the space segment is at the end of a wire (i.e., $i = 1$ or N_s for a single wire), then we may still formally use Equations (10) and (11), but with $I_{0j} = I_{N_s+1,j} \equiv 0$ and $\Delta_0 = \Delta_{N_s+1} \equiv 0$. Thus, the current on the end segments is interpolated to zero at the wire ends. Practically speaking, this has the effect, for example, of changing Equation (10) to

$$I_{1j}(s_1'', t_j'') = \sum_{l=0}^{l=1} \sum_{m=n}^{m=n+2} B_{1j}^{(l,m)} I_{1+l,j+m} \tag{10'}$$

while $B_{1j}^{(l,m)}$ is given by Eq. (11) with $s_0 \equiv s_1 - \Delta_1/2$. A similar result holds for the $i = N_s$ segment. For the case where segment N_s connects to segment 1 as in a ring, then $I_{N_s+1,j} \equiv I_{1,j}$, etc.

Approximation of the Integral Equation

We return now to the integral equation (8) and by incorporating the current expansion above with some additional manipulation, reduce it to a form convenient for numerical treatment. Let us first space-segment the structure into N_S sections of contour $\Delta\tilde{C}_i = \Delta\tilde{C}(\bar{r}_i)$ such that

$$C(\bar{r}) \approx \tilde{C}(\bar{r}) = \sum_{i=1}^{N_S} \Delta\tilde{C}_i ,$$

where the approximately equals sign is used because the segmented model and the actual structure may not exactly conform. For instance, it is advantageous to use a sequence of straight segments to model a curved wire because the s' integration over the straight segments can be analytically performed, which is not possible for general curved segments. Note that, if $\tilde{C}(\bar{r})$ is not identical to $C(\bar{r})$, the current solution for $\tilde{C}(\bar{r})$, even if exact, would only approximate the true current on $C(\bar{r})$.

The integral equation (8), when $C(\bar{r})$ is replaced by $\tilde{C}(\bar{r})$, may then be written

$$\frac{\mu_0}{4\pi} \sum_{i=1}^{N_S} \hat{s} \cdot \int_{\Delta C_i} \left[\frac{\hat{s}'}{R} \frac{\partial}{\partial t'} I(s', t') + c \frac{\bar{R}}{R^2} \frac{\partial}{\partial s'} I(s', t') - c^2 \frac{\bar{R}}{R^3} q(s', t') \right] ds'$$
$$= \hat{s} \cdot \bar{E}^A(s, t) \approx \hat{s} \cdot \bar{E}^A(s, t) \tag{12}$$
$$s \in \tilde{C}(\bar{r}), \qquad s \in C(\bar{r})$$

where all geometry-dependent quantities in (12) are evaluated along $\tilde{C}(\bar{r})$ in place of $C(\bar{r})$ and for simplicity we no longer explicitly include the wire radius displacement in Eq. (12). We can reduce (12) to the following form by using (9):

$$\frac{\mu_0}{4\pi} \sum_{i=1}^{N_S} \sum_{j=1}^{N_T} V(t' - t_j)\, \hat{s} \cdot \int_{\Delta C_i} \left[\frac{\hat{s}_i}{R_i} \frac{\partial}{\partial t''_j} I_{ij}(s''_i, t''_j) + c \frac{\bar{R}_i}{R_i^2} \frac{\partial}{\partial s''_i} I_{ij}(s''_i, t''_j) \right.$$
$$\left. - c^2 \frac{\bar{R}_i}{R_i^3} q_{ij}(s''_i, t''_j) \right] ds''_i \simeq \hat{s} \cdot \bar{E}^A(s, t) \qquad s \in \tilde{C}(\bar{r}) \tag{13}$$

where

$$R_i \equiv R_i(s'') = | \bar{r} - \bar{r}_i - \bar{s}_i |$$
$$\bar{s}_i \equiv s'' \hat{s}_i \,.$$

Since $t' = t - R_i/c$ we see that at time t only the current and charge in the time interval centered at t_v, where $| t - R_i/c - t_v | \leqslant \delta_v/2$, will contribute to the field at s and t.

The final step in reducing the integral equation to a form suitable for numerical solution involves specification of the way in which the two sides of the integral equation (13) are to be numerically related. An exact solution of the original integral equation (8) would result in zero tangential electric field everywhere on the structure for all time and for any $\bar{E}^A$ specified. Since the current expansion (9) is only approximate, the boundary condition on tangential $\bar{E}$, or equivalently, agreement between the left and right sides of (13), can only be approximately obtained. The simplest and most obvious condition is that (13) must be exactly true at the centers of the N_S space segments and N_T time intervals. This establishes a total of $N_S N_T$ relationships that can be expected to suffice for obtaining the $N_S N_T$ constants I_{ij} needed to determine the current.

Imposition of this boundary condition can be achieved, in the method of moments, by the use of delta-function weights [15, 16]. In other words, the integral equation is multiplied by $\delta(s - s_u)\, \delta(t - t_v)$ $[u = 1,..., N_S\,;\ v = 1,..., N_T]$ and integrated over $\tilde{C}(\bar{r})$ and all time. The result is that (13) can be written

$$\frac{\mu_0}{4\pi} \sum_{i=1}^{N_S} \hat{s}_u \cdot \int_{-\Delta_i/2}^{\Delta_i/2} \left[\frac{\hat{s}_i}{R_{iu}} \frac{\partial}{\partial t''_j} I_{ij}(s''_i, t''_j) + c \frac{\bar{R}_{iu}}{R_{iu}^2} \frac{\partial}{\partial s''_i} I_{ij}(s''_i, t''_j) \right.$$
$$\left. - c^2 \frac{\bar{R}_{iu}}{R_{iu}^3} q_{ij}(s''_i, t''_j) \right]_{t_j = t_v - R_{iu}/c - t''_j} ds''_i = \hat{s}_u \cdot \bar{E}^A(s_u\,, t_v);$$
$$u = 1,..., N_S\,, \quad v = 1,..., N_T\,, \tag{14}$$

where

$$R_{iu} = | \bar{r}_u - \bar{r}_i - \bar{s}_i |$$

and with

$$t_j'' = t_v - R_{iu}/c - t_j ,$$

where j is selected such that $| t_j'' | \leqslant \delta_j/2$.

Reduction of the Integral Equation to Linear System Form

Equation (14) is in a form suitable for calculation of the I_{ij} current samples. It does not, however, very clearly demonstrate the relationship that exists between the various I_{ij}. That requires, in general, the solution of a sparse structure-geometry-dependent linear system (for equal time steps) prior to a stepwise solution of (14) as a function of t_v. It is thus useful to introduce the explicit form of $I_{ij}(s_i'', t_j'')$ into (14), as well as to rewrite $q_{ij}(s_i'', t_j'')$ in terms of the current.

In most cases, equal time steps are usable, so that with $\delta_j = \delta$ for all j and counting time from $t = 0$ we have

$$j = v - R_{iu}/c\delta \equiv v - t_{iu} , \tag{15}$$

where $R_{iu}/c\delta$ is rounded off to the closest integer value r_{iu}.

Furthermore, the time and space derivatives of $I_{ij}(s_i'', t_j'')$ can be written as

$$\frac{\partial}{\partial t_j''} I_{ij}(s_i'', t_j'') = \sum_{l=-1}^{l=1} \sum_{m=+n}^{m=n+2} {}_tB_i^{(l,m)} I_{i+l,j+m} \tag{16}$$

and

$$\frac{\partial}{\partial s_i''} I_{ij}(s_i'', t_j'') = \sum_{l=-1}^{l=1} \sum_{m=+n}^{m=n+2} {}_sB_i^{(l,m)} I_{i+l,j+m} , \tag{17}$$

where

$${}_tB_i^{(l,m)} = \sum_{w=+n}^{w=n+2}{}^{m} \frac{1}{(t' - t_{j+w})} B_{ij}^{(l,m)} = \sum_{w=n}^{w=n+2}{}^{m} \frac{1}{(t_j'' + t_j - t_{j+w})} B_{ij}^{(l,m)}$$

$${}_sB_i^{(l,m)} = \sum_{r=-1}^{r=1}{}^{l} \frac{1}{(s' - s_{i+r})} B_{ij}^{(l,m)} = \sum_{r=-1}^{r=+1}{}^{l} \frac{1}{(s_i'' + s_i - s_{i+r})} B_{ij}^{(l,m)}.$$

Note that $B_{ij}^{(l,m)}$ is independent of the specific time step for equal δ_j's and becomes

$$B_i^{(l,m)} = \prod_{p=-1}^{p=+1}{}^{l} \prod_{q=n}^{q=n+2}{}^{m} \frac{(s_i'' + s_i - s_{i+p})}{(s_{i+l} - s_{i+p})} \frac{(t'' - q\delta)}{(m - q)\,\delta} ,$$

where, since $-\delta/2 \leqslant t_j'' \leqslant \delta/2$, the subscript is omitted from t''.

The charge term in (14) may be obtained from integrating the divergence of the current; i.e.,

$$q(s', t') = -\int_{-\infty}^{t'} \frac{\partial}{\partial s'} I(s', \tau)\, d\tau.$$

The charge at time step t_j may be seen to include all sampled current values up to t_j. It is convenient to define

$$q_{ij} = -\sum_{k=1}^{k=j} \frac{\partial}{\partial s_i''} \int_{-\delta/2}^{\delta/2} I_{ik}(s_i'', t'')\, dt'',$$

from which

$$q_{ij}(s_i'', t_j'') = \sum_{l=-1}^{l=+1} \sum_{m=n}^{m=n+2} B_{ij}^{(l,m)} q_{i+l,j+m}\,.$$

Then, by defining the auxiliary quantity (see Appendix A for an alternative treatment)

$$Q_{ij} = -\frac{\partial}{\partial s_i''} \int_{-\delta/2}^{\delta/2} I_{ij}(s_i'', t'')\, dt''$$

we see that

$$q_{ij} = q_{i,j-1} + Q_{ij} = q_{i,j-2} + Q_{ij} + Q_{i,j-1} = \sum_{s=1}^{j} Q_{is}\,.$$

In terms of the I_{ij} current samples, Q_{ij} becomes

$$Q_{ij} = \sum_{l=-1}^{l=+1} \sum_{m=n}^{m=n+2} C_i^{(l,m)} I_{i+l,j+m}\,,$$

where

$$C_i^{(l,m)} = \left(-\delta \sum_{r=-1}^{r=+1}{}^{l} \frac{1}{(s_i - s_{i+r})} \prod_{p=-1}^{p=+1}{}^{l} \frac{(s_i - s_{i+p})}{(s_{i+l} - s_{i+p})}\right)\left(\sum_{s=0}^{s=1} \prod_{q=n}^{q=n+2}{}^{m} \frac{q^s}{(m-q)}\right).$$

~~The spacetime dependent charge can then be written as~~

$$q_{ij}(s_i'', t'') = \sum_{l=-1}^{+1} \sum_{m=n'}^{n'+2} B_i^{(l,m)} \sum_{s=1}^{j+m} \sum_{r=-1}^{1} \sum_{t=n}^{n+2} C_{i+l}^{(r,t)} I_{i+l+r,s+t}\,, \tag{18}$$

where $n' = -2$ for $R_{iu}/c\delta < 1.5$ and -1 otherwise has been used in place of n to avoid obtaining current terms later than time-step v. The $B_i^{(l,m)}$ in (18) is also defined in terms of n' in place of n in Eq. (11).

Substitution of (16) through (18) into the integral equation (14) allows an expression for the unknown currents at time step t_v to be obtained. It is

$$\frac{\mu_0}{4\pi}\sum_{i=1}^{N_S}\hat{s}_u\cdot\int_{-\Delta_i/2}^{\Delta_i/2}\Bigg\{\Bigg[\frac{\hat{s}_i}{R_{iu}}\sum_{l=-1}^{l=+1}\sum_{m=n}^{m=n+2}{}_tB_i^{(l,m)}+c\,\frac{\bar{R}_{iu}}{R_{iu}^2}\sum_{l=-1}^{l=+1}\sum_{m=n}^{m=n+2}{}_sB_i^{(l,m)}\Bigg]I_{i+l,j+m}$$
$$-c^2\,\frac{\bar{R}_{iu}}{R_{iu}^3}\sum_{l=-1}^{l=+1}\sum_{m=n'}^{m=n'+2}B_i^{(l,m)}\sum_{s=1}^{j+m}\sum_{r=-1}^{r=+1}\sum_{t=n}^{t=n+2}C_{i+l}^{(r,t)}I_{i+l+r,s+t}\Bigg\}\,ds_i''$$
$$=-\hat{s}_u\cdot\bar{E}^A(s_u,t_v). \tag{19}$$

where $I_{i,j}=\Delta_i\equiv 0$ for $i,j\leqslant 0$ or $i>N_s$.

A rearrangement of (19) allows the current $I_{i,v}$ to be specified finally in terms of $E_{u,v}^A=-\hat{s}_u\cdot\bar{E}^A(s_u,t_v)$ and the known current values $I_{i,j}$, $j=1,\ldots,v-1$ as

$$\frac{\mu_0}{4\pi}\sum_{i=1}^{N_S}\sum_{l=-1}^{l=+1}\hat{s}_u\cdot\int_{-\Delta_i/2}^{\Delta_i/2}\Bigg\{\sum_{p=0}^{1}\delta(r_{iu}-p)\left(\frac{\hat{s}_i}{R_{iu}}\,{}_tB_i^{(l,p)}+c\,\frac{\bar{R}_{iu}}{R_{iu}^2}\,{}_sB_i^{(l,p)}\right)I_{i+l,v}$$
$$-c^2\,\frac{\bar{R}_{iu}}{R_{iu}^3}\sum_{q=0}^{q=2}\delta(r_{iu}-q)\,B_i^{(l,\langle q/2\rangle)}\sum_{r=-1}^{+1}C_{i+l}^{(r,\langle\frac{q+1}{2}\rangle)}I_{i+l+r,v}\Bigg\}\,ds_i''$$
$$=E_{uv}^A-\frac{\mu_0}{4\pi}\sum_{i=1}^{N_S}\sum_{l=-1}^{1}\hat{s}_u\cdot\int_{-\Delta_i/2}^{\Delta_i/2}\Bigg\{\sum_{m=n}^{n+2}{}^{r_{iu}}\left(\frac{\hat{s}_i}{R_{iu}}\,{}_tB_i^{(l,m)}+c\,\frac{\bar{R}_{iu}}{R_{iu}^2}\,{}_sB_i^{(l,m)}\right)I_{i+l,v-r_{iu}+m}$$
$$-c^2\,\frac{\bar{R}_{iu}}{R_{iu}^3}\sum_{m=n'}^{n'+2}B_i^{(l,m)}\sum_{s=1}^{v-r_{iu}+m}\sum_{r=-1}^{+1}\sum_{t=n}^{n+2}{}^{\langle\frac{r_{iu}+|m|+1}{2}\rangle}C_{i+l}^{(r,t)}I_{i+l+r,s+t}\Bigg\}\,ds_i';$$
$$v=1,\ldots,N_S,\quad v=1,\ldots,N_T, \tag{20}$$

where $\langle\ \rangle$ signifies rounding off to the nearest lower integral value of the quotient.

Equation (20), while appearing rather complicated, may be seen to relate the current at time-step v to previous current values through a series many of whose coefficients are zero. A matrix representation clearly illustrates the sparseness of the matrix representing the left side of the equation. It is then obvious that by starting at time-step $v=1$, prior to which all quantities are zero by definition, it is possible to obtain I_{i1}, I_{i2}, etc., by a repeated sequence of matrix operations. Since it is further known that the matrix elements are time-independent, a single inversion of the sparse matrix on the left side of (20) is required.

It is convenient to rewrite (2) in matrix form as

$$\bar{\bar{Z}} \cdot \bar{I}_v = \bar{E}_v + \sum_{l=-1}^{+1} \sum_{m=n}^{n+2} \sum^{r_{i-l,u}} \bar{\bar{X}}^{(l,m)} \cdot \bar{I}_{v-r_{i-l,u}+m}$$

$$+ \sum_{l=-1}^{+1} \sum_{r=-1}^{+1} \sum_{m=n'}^{n'+2} \sum_{t=n}^{n+2} \left\langle \frac{r_{i-l-r,u}+|m|+1}{2} \right\rangle \bar{\bar{W}}^{(l,m,r,t)}$$

$$\times \sum_{s=1}^{v-r_{i-l-r,u}+m} \bar{I}_{v-r_{i-l-r,u}+m+t+1-s}\,, \tag{21}$$

where

$$Z_{ui} = \frac{\mu_0}{4\pi} \hat{s}_u \cdot \sum_{l=-1}^{+1} \left\{ \int_{\Delta_{i-l}/2}^{\Delta_{i-l}/2} ds''_{i-l} \sum_{p=0}^{1} \delta(r_{i-l,u} - p) \left[\frac{\hat{s}_{i-l}}{R_{i-l,u}}\, {}_tB_{i-l}^{(l,p)} + c\, \frac{\bar{R}_{i-l,u}}{R^2_{i-l,u}}\, {}_sB_{i-l}^{(l,p)} \right] \right.$$

$$\left. - c^2 \sum_{r=-1}^{+1} \int_{\Delta_{i-l-r}/2}^{\Delta_{i-l-r}/2} ds''_{i-l-r}\, \frac{\bar{R}_{i-l-r,u}}{R^3_{i-l-r,u}} \sum_{q=0}^{2} \delta(r_{i-l-r,u} - q)\, B_{i-l-r}^{(l,\langle q/2\rangle)} C_{i-r}^{\left(r,\left\langle\frac{q+1}{2}\right\rangle\right)} \right\}$$

and

$$X_{ui}^{(l,m)} = -\frac{\mu_0}{4\pi} \hat{s}_u \cdot \left\{ \int_{-\Delta_{i-l}/2}^{\Delta_{i-l}/2} ds''_{i-l} \left[\frac{\hat{s}_{i-l}}{R_{i-l,u}}\, {}_tB_{i-l}^{(l,m)} + c\, \frac{\bar{R}_{i-l,u}}{R^2_{i-l,u}}\, {}_sB_{i-l}^{(l,m)} \right] \right\}$$

$$W_{ui}^{(l,m,r,t)} = c^2 \frac{\mu_0}{4\pi} \hat{s}_u \cdot \int_{-\Delta_{i-l-r}/2}^{\Delta_{i-l-r}/2} ds''_{i-l-r}\, \frac{\bar{R}_{i-l-r,u}}{R^3_{i-l-r,u}}\, B_{i-l-r}^{(l,m)} C_{i+l}^{(r,t)}$$

and

$$\bar{I}_v = \begin{bmatrix} I_{1v} \\ I_{2v} \\ \vdots \\ I_{N_S v} \end{bmatrix}.$$

The current can thus be obtained from

$$\bar{I}_v = \bar{\bar{Y}} \cdot \left\{ \bar{E}_v + \sum_{l=-1}^{+1} \sum_{m=n}^{n+2} \sum^{r_{i-l,u}} \bar{\bar{X}}^{(l,m)} \cdot \bar{I}_{v-r_{i-l,u}+m} \right.$$

$$\left. + \sum_{l=-1}^{+1} \sum_{r=-1}^{+1} \sum_{m=n'}^{n'+2} \sum_{t=n}^{n+2} \left\langle \frac{r_{i-l-r,u}+|m|+1}{2} \right\rangle \bar{\bar{W}}^{(l,m,r,t)} \sum_{s=1}^{v-r_{i-l-r,u}+m} \bar{I}_{v-r_{i-l-r,u}+m+t-s+1} \right\};$$

$$v = 1,\ldots, N_T\,, \tag{22}$$

where $\bar{\bar{Y}} = [\bar{\bar{Z}}]^{-1}$.

Since all the matrix elements are time-independent, the matrix operations indicated in (22) can be performed, and their results stored, prior to actual evaluation of $\bar{I}_v$ as a function of time. All that is necessary then is repeated multiplication of these stored matrices by the time-dependent field and the already-known current values.

Equation (22) represents a formal (but approximate) solution of the integral equation (14) for the time-dependent current excited on a wire structure by a time-varying electric field source $\bar{E}^A(\bar{r}, t)$. While $\bar{E}^A(\bar{r}, t)$ may in principle be an arbitrarily varying function of time, it is convenient to represent it as an approximate impulse (Gaussian) or an approximate step function from which the scatterer or antenna response to other field variations may be obtained from convolution. In this sense the solution is independent of the temporal variation of the source.

Note that a solution for I_{ij} from (22) depends, however, on the geometric or spatial nature of the exciting field. Thus in contrast to the frequency-domain case, whose solution is independent of the particular source spatial variation, the time-domain solution is source-geometry dependent. These factors, of course, influence the relative efficiencies of these two approaches for the solution of a particular problem, a topic considered further below.

Comparison of Time-Domain and Frequency-Domain Analyses

In order to establish guidelines for the relative utility of either frequency or time domain analysis, the time requirements for specific computations in each domain will be presented for calculations on a Control Data 6600 computer. The computation times using other computers will naturally be different but the relative advantages of analysis in one domain over another will nonetheless be illustrated. The total number of samples in either space (N_S), time (N_T), or frequency (N_F), required for accurate results in either domain will not be dealt with here but is relegated to the discussion in Part II of this paper [17]. Rather, in this analysis we will allow the number of space and time samples to be parameters. Furthermore, we will assume that all response temporal waveforms and their spectra have their energy primarily contained within finite intervals of length T and F, respectively. Then we have, $N_T = T/\Delta T$ and $N_F = F/\Delta F$, and, from the Shannon–Kotelnikov Sampling Theorem which specifies the relationship between the temporal step size and the maximum frequency of the band limited spectrum, we have $F = 1/2\,\Delta T$. Since we are dealing with real functions of time and are concerned with spectra on $(0, F)$, we also have $T = 1/\Delta F$, and as a result, $N_T = 2N_F$. We will now proceed to investigate the relative advantages and disadvantages of the frequency domain and time domain approaches when one is trying to determine the spectral response over the interval $(0, F)$.

As has been previously pointed out, a moment-method solution of the frequency-domain integral equation, while perhaps requiring a matrix factorization or inversion for a given frequency, is independent of exciting source geometry. Hence, induced currents at a particular frequency can be determined for any illuminating field. Having once solved the integral equation of the form $[E^A] = [Z][I]$, one merely retains the inverse matrix $[Z]^{-1}$, which is only structure-geometry and frequency dependent. Subsequent evaluations of the current distribution for various source configurations at a given frequency then require the matrix multiplication $[I] = [Z]^{-1}[E^A]$. On the other hand, the time-domain solution is dependent on source geometry and hence requires the entire solution process to be repeated for each different source configuration. In the following timing considerations we therefore take the most general approach and consider the situation where a solution is required for M different source configurations.

For the multiple source problem, the required time for computing the time history of induced current on the structure has been found to be approximately

$$S_{Tc}(\text{sec}) = 1.3 \times 10^{-4} M N_S^2 N_T .$$

Once the currents are known, finding the time history of the radiated field at N_A observation angle requires the additional time

$$S_{Tf}(\text{sec}) = 2 \times 10^{-5} M N_S N_T N_A .$$

The numerical coefficient values used above have been obtained from actual calculations on the CDC-6600 computer. Note that in general $S_{Tf} \ll S_{Tc}$. Since the Fast Fourier Transform (FFT) allows the frequency response to be obtained with no significant increase over the times indicated above, it is not explicitly considered here.

In order to find the spectrum of current behavior using frequency domain calculations, one needs to perform the calculation at N_F frequencies such that the spectral response is adequately represented. Hence the total amount of time to determine a given frequency response is N_F multiplied by the computation time for a single frequency. Computer experimentation has shown that the determination of the induced current at N_F frequencies using moment methods with sinusoidal interpolation (Poggio and Miller, 1971) requires

$$S_{Fc}(\text{sec}) = 2 \times 10^{-3} N_S^2 N_F + 5 \times 10^{-6} N_S^3 N_F + 2 \times 10^{-5} N_S^2 M N_F$$

where the first term pertains to matrix fill time, the second to matrix inversion time, and the last to the current evaluation time. Note that for multiple source configurations only the current evalutation time is multiplied by M. The time required for the evaluation of the radiated field at N_A angles having already found the current given by

$$S_{Ff}(\text{sec}) = 3 \times 10^{-4} M N_S N_F N_A$$

Again we find that $S_{Ff} \ll S_{Fc}$.

The ratio of the computation times required to obtain the time history or the frequency spectra of the current from the frequency domain compared with the time domain is therefore

$$(S_{Fc}/S_{Tc}) = .077 + (7.7 + 1.9 \times 10^{-2} N_S)/M$$

For $N_S < 100$ this ratio reduces to

$$(S_{Fc}/S_{Tc}) \approx .077 + 7.7/M$$

The advantage of the time domain over the frequency domain for a fixed source geometry ($M = 1$) is clearly exhibited by the preceding ratio. However, as M increases the advantage decreases. For the current calculation, the ratio is reduced to unity when $M \approx 8.4 + 2.1 \times 10^{-2} N_S$.

The ratio derived above is merely a crude guideline. It must be remembered that N_S was assumed constant over the entire frequency range for the frequency domain calculation. By writing N_S as a function of frequency one could sum over the required frequencies but this has been found to lead to a reduction by only a factor of about one-third. This reduction would tend to make the frequency domain somewhat more competitive with the time domain when one wishes to determine frequency response characteristics. There is, however, a feature of time domain response which we have not taken into account. As will be evident in Part II of this paper, the time domain response normally exhibits a "ringing" behavior, e.g., oscillations of the form $e^{-\alpha t} \sin \omega t$ for the current at late times. This tends to shorten the time record which must be computed since extrapolation leads to accurate results beyond some time T_U. The required number of time steps N_T is therefore reduced for the time domain calculation thereby leading to a corresponding decrease in the computation time. Factors of this type and others which are dependent on the particular structure being analyzed have not been introduced.

One can surmise from the preceding discussions that the time domain approach is more efficient for calculating spectral responses (or time histories) when one is concerned with bistatic radar cross-section or antenna radiation patterns for a few source configurations. On the other hand, the frequency domain approach is more efficient when monostatic radar cross-section for many directions of incidence or antenna patterns for many source configurations are of interest.

TABLE I

Frequency Domain vs Time Domain

	Frequency domain	Time domain	Relative advantage FD/TD
Multiple Source			
Monochromatic	$(L/\lambda)^3$	$(L/\lambda)^4$	$\sim(L/\lambda)^{-1}$
Transient	$(L/\lambda)^4$	$(L/\lambda)^4$	~ 1
Single Source			
Monochromatic	$(L/\lambda)^3$	$(L/\lambda)^3$	~ 1
Transient	$(L/\lambda)^4$	$(L/\lambda)^3$	$\sim(L/\lambda)$

The above results can be conveniently summarized as to problem type and application as shown in Table I. There the highest order dependence of the calculation time upon structure size in wavelengths is exhibited as a function of source geometry, response desired (monochromatic or transient) and approach used. In obtaining these results, we assume that N_S, N_F, N_A, and M are all proportional to L/λ. The actual ratios of frequency to time domain calculation times depend of course upon the coefficients which accompany these terms, hence the fact that for example, $S_{Fc}/S_{Tc} \sim 8$ for $M = 1$ with $N_S < 100$, while Table I indicates that for this case $S_{Fc}/S_{Tc} \sim L/\lambda$. This demonstrates that comparisons based on highest order terms can be potentially misleading, yet factually correct, if their respective coefficients are ignored.

Impedance Matrix Elements

Before concluding our discussion of the numerical analysis of the integral equation (9), we should comment on the calculation of the coefficients that appear in the various (impedance) matrices Z, X, and W. It is clear from (7), (16), (17), and (21) and since $t' = t - R/c$ that all the interaction matrix elements have variations of the form $s_i''^a/R_{iu}^b(s_i'')$, where $a, b = 0,..., 3$.

Since

$$\bar{R}_{iu}(s_i'') = \bar{r}_u - \bar{r}_i - \bar{s}_i$$
$$= \bar{R}_{iu} - s_i''\hat{s}_i\,,$$

then

$$R_{iu}(Rs_i'') = [\bar{R}_{iu} \cdot \bar{R}_{iu} - 2s_i''\hat{s}_i \cdot \bar{R}_{iu} + s_i''^2]^{1/2}$$

and we thus have

$$I_{iu}^{ab} = \int_{-\Delta_i/2}^{\Delta_i/2} \frac{s_i''^a \, ds_i''}{(R_{iu}^2 - 2s_i''\hat{s}_i \cdot \bar{R}_{iu} + s_i''^2)^{b/2}} \tag{23}$$

as the integrals to be evaluated, where it is understood that the thin-wire approximation requires $R_{ii}^2(s_i'') = a_i^2 + s_i''^2$, with a_i the wire radius of segment i.

Integrals of the form (23) are fortunately analytically integrable. Computation of the impedance matrix elements is thus relatively efficient. Precautions are required, however, to ensure that the analytical form used for the integral evaluation is appropriate for the integration limits involved; e.g., subtraction of large, nearly equal numbers should be avoided. The actual forms used for the impedance element calculation were validated by a rather extensive comparison with numerically integrated values of I_{iu}^{ab}. A summary of the integral forms used is given in Appendix *B*, together with the final form of the integral equation in terms of the I_{iu}^{ab}.

Conclusions

A technique has been described for obtaining time-domain solutions for radiation and scattering problems associated with wire structures. The time-dependent electric-field integral equation pertinent to these structures has been developed and, using subsectional collocation, has been reduced to a form suitable for numerical computation. A two-dimensional Lagrangian interpolation function of order three in each dimension has been used and appropriate expressions developed for the approximate form of the integral equation. A discussion of the relative advantages of time-domain and frequency-domain calculations has been included to illustrate the advantages of time-domain calculations in certain situations. Typical results obtained using the techniques presented in this paper for wire scatterers and antennas wil be presented in Part II of this paper.

APPENDIX A: Alternative Interpolation Scheme

As an alternative to the definition for Q_{ij} we could instead use

$$Q_{ij} = - \int_{-\delta/2}^{\delta/2} I_{ij}(s_i'', t'') \, dt'' \Big|_{s_i''=0}$$

and subsequently then obtain

$$Q_{ij}(s_i'', t'') = \frac{\partial}{\partial s_i''} \sum_{l=-1}^{+1} \sum_{m=n}^{n+2} B_i^{(l,m)} \sum_{s=1}^{j+m} Q_{i+l,s}$$

$$= \sum_{l=-1}^{+1} \sum_{m=n'}^{n+2} {}_sB_i^{(l,m)} \sum_{s=1}^{j+m} \sum_{r=-1}^{+1} \sum_{t=n}^{n+2} D_i^{(r,t)} I_{i+l+r,s+t} ,$$

where

$$D_i^{(l,m)} = -\delta \prod_{p=-1}^{+1}{}^{l} \frac{(s_i - s_{i+p})}{(s_{i+l} - s_{i+p})} \sum_{s=0}^{+1} \prod_{q=n}^{n+2}{}^{m} \frac{q^s}{(m-q)} .$$

This interpolation form for the charge modifies only the $\bar{G}_{i,u,l,m,r,t}$ function that appears in the charge term of the integral equation (see Equation B-1). It has been found to provide greater numerical stability than (18) in some cases.

APPENDIX B: Explicit Impedance Matrix Coefficients

The integral equation (21) may be expressed in terms of the integral forms I_{iu}^{ab} and various products of the interpolation coefficients. It may be established that, for example

$$Z_{ui} = \frac{\mu_0}{4\pi} \hat{s}_u \cdot \sum_{l=-1}^{1} \sum_{a=0}^{3} \sum_{b=0}^{3} \Big[\sum_{p=0}^{1} \delta(r_{i-l,u} - p) \bar{F}_{i-l,u,l,p}^{(a,b)} I_{i-l,u}^{(a,b)}$$

$$+ \sum_{r=-1}^{+1} \sum_{q=0}^{2} \delta(r_{i-l-r,u} - q)\, \dot{\bar{G}}_{i-l-r,u,l,\langle q/2\rangle,r,\langle (q+1)/2\rangle}^{(a,b)} I_{i-l-r,u}^{(a,b)} \Big], \qquad \text{(B-1)}$$

where

$$I_{i,u}^{(a,b)} = \int_{-\Delta_i/2}^{\Delta_i/2} ds_i'' \frac{s_i''^a}{R_{iu}^b(s_i'')}$$

and

$$\bar{F}_{i,u,l,m}^{(0,0)} = -\frac{1}{D_i^{(l,m)} c} \Big\{ \hat{s}_i\, 2 \prod_{p=-1}^{1}{}^{l} (s_i - s_{i+p}) + \bar{R}_{iu} \sum_{p=-1}^{1}{}^{l} (s_i - s_{i+p}) \Big\}$$

$$\bar{F}_{i,u,l,m}^{(1,0)} = -\frac{1}{D_i^{(l,m)} c} \Big\{ \hat{s}_i\, 2 \sum_{p=-1}^{1}{}^{l} (s_i - s_{i+p}) + \Big[\sum_{p=-1}^{1}{}^{l} (s_i - s_{i+p})\, \hat{s}_i - 2\bar{R}_{iu} \Big] \Big\}$$

$$\bar{F}^{(2,0)}_{i,u,l,m} = -\frac{4}{D_i^{(l,m)}}\frac{\hat{s}_i}{c}$$

$$F^{(3,0)}_{i,u,l,m} = 0$$

$$F^{(0,1)}_{i,u,l,m} = \delta\frac{1}{D_i^{(l,m)}}\left\{\hat{s}_i\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\, q\right]\prod_{p=-1}^{1}{}^{l}\,(s_i - s_{i+p})\right.$$
$$\left. - \bar{R}_{iu}\sum_{p=-1}^{1}{}^{l}\,(s_i - s_{i+p})\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\, q\right]\right\}$$

$$F^{(1,1)}_{i,u,l,m} = \delta\frac{1}{D_i^{(l,m)}}\left\{\hat{s}_i\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\, q\right]\right.$$
$$\left. - \left[2\bar{R}_{iu} - \hat{s}_i\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right]\left[-2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\, q\right]\right\}$$

$$F^{(2,1)}_{i,u,l,m} = 3\delta\frac{1}{D_i^{(l,m)}}\left\{\hat{s}_i\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\, q\right]\right\}$$

$$F^{(3,1)}_{i,u,l,m} = 0$$

$$F^{(0,2)}_{i,u,l,m} = \frac{1}{D_i^{(l,m)}}\, c\bar{R}_{iu}\,\delta^2\left[r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\, q + \prod_{q=n}^{n+2}{}^{m}\, q\right]$$

$$F^{(1,2)}_{i,u,l,m} = \delta^2\frac{1}{D_i^{(l,m)}}\left\{c\left[2\bar{R}_{iu} - \hat{s}_i\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right]\right.$$
$$\left.\times\left[r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\, q + \prod_{q=n}^{n+2}{}^{m}\, q\right]\right\}$$

$$F^{(2,2)}_{i,u,l,m} = -2c\delta^2\frac{1}{D_i^{(l,m)}}\left\{r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\, q + \prod_{q=n}^{n+2}{}^{m}\, q\right\}$$

$$F^{(3,2)}_{i,u,l,m} = 0$$

$$F^{(3,b)}_{i,u,l,m} = 0 \qquad [\text{all } b]$$

$$G^{(a,0)}_{i,u,l,m} = 0 \qquad [\text{all } a]$$

$$\bar{G}^{(0,1)}_{i,u,l,m,r,t} = -\frac{1}{D_i^{(l,m)}}\bar{R}_{iu}\prod_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\, C^{(r,t)}_{i+l}$$

$$\bar{G}^{(1,1)}_{i,u,l,m,r,t} = -\frac{1}{D_i^{(l,m)}}\left[\bar{R}_{iu}\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p}) - \hat{s}_i\prod_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right] C_{i+l}^{(r,t)}$$

$$\bar{G}^{(2,1)}_{i,u,l,m,r,t} = -\frac{1}{D_i^{(l,m)}}\left[\bar{R}_{iu} - \hat{s}_i\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right] C_{i+l}^{(r,t)}$$

$$\bar{G}^{(3,1)}_{i,u,l,m,r,t} = \frac{1}{D_i^{(l,m)}}\,\hat{s}_i C_{i+l}^{(r,t)}$$

$$\bar{G}^{(0,2)}_{i,u,l,m,r,t} = \delta c\,\frac{1}{D_i^{(l,m)}}\,\bar{R}_{iu}\prod_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\,q\right] C_{i+l}^{(r,t)}$$

$$\bar{G}^{(1,2)}_{i,u,l,m,r,t} = \delta c\,\frac{1}{D_i^{(l,m)}}\left\{\left[\bar{R}_{iu}\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p}) - \hat{s}_i\prod_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right]\right.$$

$$\left.\times\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\,q\right]\right\} C_{i+l}^{(r,t)}$$

$$\bar{G}^{(2,2)}_{i,u,l,m,r,t} = c\delta\,\frac{1}{D_i^{(l,m)}}\left\{\left[\bar{R}_{iu} - \hat{s}_i\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right]\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\,q\right]\right\} C_{i+l}^{(r,t)}$$

$$\bar{G}^{(3,2)}_{i,u,l,m,r,t} = -c\delta\,\frac{1}{D_i^{(l,m)}}\,\hat{s}_i\left[2r_{iu} - \sum_{q=n}^{n+2}{}^{m}\,q\right] C_{i+l}^{(r,t)}$$

$$G^{(0,3)}_{i,u,l,m,r,t} = -c^2\delta^2\,\frac{1}{D_i^{(l,m)}}\left\{\bar{R}_{iu}\prod_{p=-1}^{1}{}^{l}\,(s_i - s_{i+p})\left[r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\,q + \prod_{q=n}^{n+2}{}^{m}\,q\right]\right\} C_{i+l}^{(r,t)}$$

$$G^{(1,3)}_{i,u,l,m,r,t} = -c^2\delta^2\,\frac{1}{D_i^{(l,m)}}\,\hat{s}_i\prod_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\left[r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\,q + \prod_{q=n}^{n+2}{}^{m}\,q\right] C_{i+l}^{(r,t)}$$

$$G^{(2,3)}_{i,u,l,m,r,t} = -c^2\delta^2\,\frac{1}{D_i^{(l,m)}}\left[\bar{R}_{iu} - \hat{s}_i\sum_{p=-1}^{+1}{}^{l}\,(s_i - s_{i+p})\right]$$

$$\times\left[r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\,q + \prod_{q=n}^{n+2}{}^{m}\,q\right] C_{i+l}^{(r,t)}$$

$$G^{(2,3)}_{i,u,l,m,r,t} = c^2\delta^2\,\frac{1}{D_i^{(l,m)}}\,\hat{s}_i\left[r_{iu}^2 - r_{iu}\sum_{q=n}^{n+2}{}^{m}\,q + \prod_{q=n}^{n+2}{}^{m}\,q\right] C_{i+l}^{(r,t)},$$

where

$$D_i^{(l,m)} = \delta^2 \prod_{p=-1}^{+1}{}^{l} (s_{i+l} - s_{i+p}) \prod_{q=n}^{n+2}{}^{m} (m - q).$$

Also,

$$X_{ui}^{(l,m)} = -\frac{\mu_0}{4\pi} \hat{s}_u \cdot \sum_{a=0}^{3} \sum_{b=0}^{3} \bar{F}_{i-l,u,l,m}^{(a,b)} \bar{I}_{i-l,u}^{(a,b)}$$

$$W_{ui}^{(l,m,r,t)} = c^2 \frac{\mu_0}{4\pi} \hat{s}_u \cdot \sum_{a=0}^{3} \sum_{b=0}^{3} \bar{G}_{i-l-r,u,l,m,r,t} I_{i-l-r,u}^{(a,b)} \cdot$$

Finally, upon letting $b = -2\hat{s}_i \cdot \bar{R}_{iu}$ and $c = R_{iu}^2 = \bar{R}_{iu} \cdot \bar{R}_{iu}$ we have

$$I_{iu}^{(0,0)} = \Delta_i$$

$$I_{iu}^{(1,0)} = 0$$

$$I_{iu}^{(2,0)} = \Delta_i^3/12$$

$$I_{iu}^{(3,0)} = 0$$

$$I_{iu}^{(0,1)} = \ln[2R_{iu}(s_i'') + | 2s_i'' + b |]_{-\Delta_i/2}^{\Delta_i/2}$$

$$I_{iu}^{(1,1)} = R_{iu}(s_i'')|_{-\Delta_i/2}^{\Delta_i/2} - \tfrac{1}{2}bI_{iu}^{(0,1)}$$

$$I_{iu}^{(2,1)} = [(\tfrac{1}{2}s_i'' - \tfrac{3}{4}b)\, R_{iu}(s_i'')]_{-\Delta_i/2}^{\Delta_i/2} + \tfrac{1}{8}(3b^2 - 4c)\, I_{iu}^{(0,1)}$$

$$I_{iu}^{(3,1)} = -\tfrac{1}{3}\{\tfrac{7}{3}bI_{iu}^{(2,1)} + [2c + \tfrac{3}{4}b^2]\, I_{iu}^{(1,1)} + \tfrac{1}{2}bcI_{iu}^{(0,1)} - [R_{iu}(s_i'')\, s_i''(s_i'' + \tfrac{1}{2}b)]_{-\Delta_i/2}^{\Delta_i/2}\}$$

$$I_{iu}^{(0,2)} = \begin{cases} \dfrac{2}{(4c - b^2)^{1/2}} \tan^{-1} \dfrac{2s_i'' + b}{(4c - b^2)^{1/2}} \Bigg|_{-\Delta_i/2}^{\Delta_i/2} \\[2ex] -\dfrac{2}{2s_i'' + b} \Bigg|_{-\Delta_i/2}^{\Delta_i/2} \qquad \text{if} \quad b^2 = 4c \end{cases}$$

$$I_{iu}^{(1,2)} = \ln R_{iu}(s_i'')|_{-\Delta_i/2}^{\Delta_i/2} - \tfrac{1}{2}bI_{iu}^{(0,2)}$$

$$I_{iu}^{(2,2)} = [s_i'' - b \ln R_{iu}(s_i'')]_{-\Delta_i/2}^{\Delta_i/2} + \tfrac{1}{2}(b^2 - 2c)\, I_{iu}^{(0,2)}$$

$$I_{iu}^{(3,2)} = -cI_{iu}^{(1,2)} - bI_{iu}^{(2,2)}$$

$$I_{iu}^{(0,3)} = \begin{cases} \dfrac{4s_i'' + 2b}{(4c - b^2)\, R_{iu}(s_i'')} \Big|_{-\Delta_i/2}^{\Delta_i/2} \\ \left| \dfrac{1}{2} \left[\dfrac{1}{(s_i + \frac{1}{2}b)^2} \right]_{-\Delta_i/2}^{\Delta_i/2} \right| & \text{if } b^2 = 4c \end{cases}$$

$$I_{iu}^{(1,3)} = \begin{cases} -\left[\dfrac{2bs_i'' + 4c}{(4c - b^2)\, R_{iu}(s_i'')} \right]_{-\Delta_i/2}^{\Delta_i/2} \\ \left| \left[\dfrac{b}{4} \dfrac{1}{(s_i'' + b/2)^2} - \dfrac{1}{(s_i'' + b/2)} \right]_{-\Delta_i/2}^{\Delta_i/2} \right| & \text{if } b^2 = 4c \end{cases}$$

$$I_{iu}^{(2,3)} = I_{iu}^{(0,1)} - bI_{iu}^{(1,3)} - cI_{iu}^{(0,3)}$$

$$I_{iu}^{(3,3)} = I_{iu}^{(1,1)} - bI_{iu}^{(2,3)} - cI_{iu}^{(1,3)} .$$

The integral equation (21) is thus expressible, with the i-summation explicitly shown, as

$$\frac{\mu_0}{4\pi} \hat{s}_u \cdot \sum_{i=1}^{N_S} \sum_{l=-1}^{+1} \sum_{a=0}^{3} \sum_{b=0}^{3} \Big[\sum_{p=0}^{1} \delta(r_{i-l,u} - p)\, \bar{F}_{i-l,u,l,p}^{(a,b)} I_{i-l,u}^{(a,b)}$$

$$+ \sum_{r=-1}^{1} \sum_{q=0}^{2} \delta(r_{i-l-r,u} - q)\, \bar{G}_{i-l-r,u,l,\langle q/2\rangle,r,\langle (q+1)/2\rangle}^{(a,b)} I_{i-l-r,u}^{(a,b)} \Big] I_{i,v}$$

$$= \hat{s}_u \cdot \bar{E}_{uv}^{A} - \frac{c^2\mu_0}{4\pi} \sum_{i=1}^{N_S} \sum_{l=-1}^{1} \sum_{m=n}^{n+2}{}^{r_{i-l,u}} \sum_{a=0}^{3} \sum_{b=0}^{3} \bar{F}_{i-l,u,l,m}^{(a,b)} I_{i,v-r_{i-l,u}+m}$$

$$+ \frac{c^2\mu_0}{4\pi} \hat{s}_u \cdot \sum_{i=1}^{N_S} \sum_{l=-1}^{+1} \sum_{m=n}^{n'+2} \sum_{r=-1}^{+1} \sum_{t=n}^{n+2}{}^{\langle (r_{i-l-r,u}+|m|+1)/2\rangle} \sum_{a=1}^{3} \sum_{b=1}^{3}$$

$$\cdot\, \bar{G}_{i-l-r,u,l,m,r,t}^{(a,b)} I_{i-l-r,u}^{(a,b)} \sum_{s=1}^{v-r_{i-l-r,u}+m} I_{i,v-r_{i-l-r,u}+m+t-s+1} \, .$$

Acknowledgment

The authors greatly appreciate the typing expertise contributed by Mrs. Margaret Dixon to the preparation of this manuscript.

References

1. C. L. Bennett and W. L. Weeks, "A Technique for Computing Approximate Electromagnetic Response of Conducting Bodies," Purdue University, Lafayette, Ind., Rept. TR-EE68-11, 1968.
2. C. L. Bennett and W. L. Weeks, *IEEE Trans. Antennas Propagation* AP-18, 627 (1970).
3. E. P. Sayre and R. F. Harrington, "Transient Response of Straight Wire Scatterers and Antennas," in *Proc. 1968 IEEE Intern. Antenna Propagation Symp.*, Boston, Mass., Sept. Sept. 9-12, 1968.
4. E. P. Sayre, "Transient Response of Wire Antennas and Scatterers," Syracuse University, Syracuse, N. Y., Technical Rept. TR-69-4 (1969).
5. A. M. Auckenthaler and C. L. Bennett, *IEEE Trans. Microwave Theory and Techniques*, Vol. **MTT-19 MIT**, 892 (1971).
6. C. L. Bennett, A. M. Auckenthaler, and J. D. De Lorenzo, Transient Scattering by Three-Dimensional Surfaces with Wires, in "Proc. 1971 IEEE Intern. Antenna Propagation Symp.," Los Angeles, California, Sept. 22–24, 1971.
7. K. K. Mei, private communication, 1969.
8. A. J. Poggio, *IEEE Trans. Antennas Propagation* **AP-19** (1971), 702.
9. A. J. Poggio and E. K. Miller, "Integral Equation Solutions of Three Dimensional Scattering Problems," MB Associates, San Ramon, Calif., Technical Rept. TM-70/20 (to appear in "Numerical Techniques for Electromagnetic Boundary Value Problems" (R. Mittra, ed.), Pergamon Press, New York, 1973).
10. S. H. Lin and K. K. Mei, *IEEE Trans. Antennas Propagation* **AP-18**, 672 (1970).
11. J. A. Stratton, "Electromagnetic Theory," McGraw-Hill, New York, 1941.
12. R. W. P. King, "The Theory of Linear Antennas," Harvard University Press, Cambridge, MA, 1956.
13. K. K. Mei, *IEEE Trans. Antennas Propagation* **AP-13**, 374 (1965).
14. S. Gee, E. K. Miller, A. J. Poggio, E. S. Selden, and G. J. Burke, Computer Techniques for Electromagnetic Scattering and Radiation, in "Proc. 1971 IEEE Intern. Symp. Electromagnetic Compatibility," Philadelphia, PA, July 13–15, 1971.
15. R. F. Harrington, "Field Computation by Moment Methods," Macmillan, New York, 1968.
16. L. V. Kantorovich and V. F. Krylov, "Approximate Methods of Higher Analysis," Interscience, New York, 1964.
17. A. J. Poggio, E. K. Miller, and G. J. Burke, "An Integro-Differential Equation Technique for the Time-Domain Analysis of Thin Wire Structures. II. Numerical Results," *J. Computational Phys.* **12** (1973), to appear.